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The linear development of Görtler vortices in growing boundary layers

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The growth of Görtler vortices in boundary layers on concave walls is investigated. It is shown that for vortices of wavelength comparable to the boundary-layer thickness the appropriate linear stability equations cannot be reduced to ordinary differential equations. The partial differential equations governing the linear stability of the flow are solved numerically, and neutral stability is defined by the condition that a dimensionless energy function associated with the flow should have a maximum or minimum when plotted as a function of the downstream variable X. The position of neutral stability is found to depend on how and where the boundary layer is perturbed, so that the concept of a unique neutral curve so familiar in hydrodynamic-stability theory is not tenable in the Görtler problem, except for asymptotically small wavelengths. The results obtained are compared with previous parallel-flow theories and the small-wavelength asymptotic results of Hall (1982*a*, *b*), which are found to be reasonably accurate even for moderate values of the wavelength. The parallel-flow theories of the growth of Görtler vortices are found to be irrelevant except for the small-wavelength limit. The main deficiency of the parallel-flow theories is shown to arise from the inability of any ordinary differential approximation to the full partial differential stability equations to describe adequately the decay of the vortex at the edge of the boundary layer. This deficiency becomes intensified as the wavelength of the vortices increases and is the cause of the wide spread of the neutral curves predicted by parallel-flow theories. It is found that for a wall of constant radius of curvature a given vortex imposed on the flow can grow for at most a finite range of values of X. This result is entirely consistent with, and is explicable by the asymptotic results of, Hall (1982*a*).

1. Introduction

Our concern is with the development of a self-consistent theory to explain the growth of Görtler vortices in growing boundary layers on concave walls. Such vortices are of importance because destabilizing centrifugal forces exist for certain boundary-layer profiles of practical importance. The existence of the vortices was predicted by Görtler (1940), who presented a linear stability analysis similar to that given by Taylor (1923) for centrifugal instabilities in Couette flow. The apparent similarity between Taylor and Görtler vortices is in fact misleading; the effect of the non-parallel nature of the basic flow, in which the latter develops, is not negligible, as was assumed by Görtler.

Apart from the obvious importance of Görtler vortices in the transition process on the lower sides of aerofoils, it is thought that they are crucial in determining the efficiency of turbine blades (see e.g. Kemp 1977; Martin & Brown 1979). In both of these engineering situations the possible co-existence of Tollmien–Schlichting waves must not be overlooked. However, as a starting point for such a complex interaction problem, we shall in this paper consider only the growth of Görtler vortices in the linear regime.

In two previous papers Hall (1982a, b), hereinafter referred to as I and II, a self-consistent asymptotic description of the growth of short-wavelength Görtler vortices was given. In particular, in I it was shown that the equations governing the centrifugal instability of a boundary-layer flow are partial differential equations. The approximations of these equations discussed by Görtler (1940) and subsequent authors result in simpler ordinary differential equations, but cannot be justified. In I a formal asymptotic solution of the full partial differential equations was given for short-wavelength vortices, whilst in II the nonlinear development of the vortices was described. The crucial simplification of these papers is that the wavelength of the vortices was taken to be small compared with the boundary-layer thickness. In this limit the vortices are concentrated in an internal viscous layer which thickens as the vortices grow. The linear theory of I provides a right-hand branch for the neutral curve in the wavenumber–Görtler-number plane. In this paper we concentrate on $O(1)$ wavenumbers, which are thought to be the most likely to occur in situations of practical importance.

The present investigation shows the regime in which the asymptotic results of I are valid and confirms one of the most surprising results of that paper. We refer to the result found in I that, for a wall of constant radius of curvature, the boundary layer ultimately becomes more stable as the fluid is convected downstream. We find that a vortex of fixed wavelength is locally unstable for only a finite distance along the boundary layer for walls of constant curvature. As the fluid moves downstream, any given vortex flow ultimately decays to zero in the absence of nonlinear effects. If the curvature of the wall increases at a sufficient rate we shall see that this ultimate region of stability is never reached and the vortex flow remains unstable infinitely far downstream of the location where it is imposed on the flow.

In order to determine the stability characteristics of Görtler vortices of wavelength comparable to the boundary-layer thickness, previous investigators have solved truncated forms of the full partial differential equations governing the stability of the flow. In particular, Görtler (1940) solved a simplified system of equations essentially obtained by retaining terms equivalent to those appearing in the Taylor-vortex linear stability equations. The solution of these equations given by Görtler was incorrect, and Hämmerlin (1955) re-solved the same equations. The neutral curve obtained in this parallel-flow approximation has a minimum at zero wavenumber. In order to remedy this deficiency, numerous authors have solved various modified forms of Görtler's equations, but without exception the modified equations are all ordinary differential equations. (For a detailed review of some of these investigations see Herbert (1976).) Perhaps the most important of these calculations was the one given by Smith (1955), who appears to have been the first investigator to allow for any non-parallel effects. Thus Smith retained some of the terms associated with the non-parallel nature of the basic flow and in fact found a critical Görtler number at a finite wavenumber. However, the equations solved by Smith were ordinary differential equations, and subsequent 'improvements' to Smith's equations have produced neutral curves quite different to his neutral curve for $O(1)$ wavenumbers.

The concept of a unique neutral curve, so familiar in parallel-flow stability problems, will be shown in this paper to be untenable for the Görtler-vortex problem. We shall show that the growth or decay of a disturbance imposed on the boundary layer depends not only how it is introduced into the flow but also on its location. This result seems to be consistent with the available experimental observations. We shall further show that the total disagreement of previous theories at $O(1)$ wavenumbers is not surprising, since, without the partial derivatives of the disturbance in the flow direction, the decay of the disturbance at the edge of the boundary layer is not described correctly. In previous investigations this decay is facilitated by a balance of diffusion of vorticity in the two directions normal to the flow direction. We show here that the appropriate balance is in fact one between convection in the flow direction and diffusion normal to the wall. Thus at small wavenumbers the vortices are still confined to the boundary layer, and the stability equations derived on this assumption remain valid. Clearly any approximation that results in an ordinary differential system must be in error for $O(1)$ wavenumbers (or less) because the decay of the vortex at the edge of the boundary layer is quite different from that predicted by the full equations.

The first step in our calculation is to eliminate the pressure and spanwise velocity component from the linear perturbation equations. The perturbation is taken to be steady and periodic in the spanwise direction with fixed wavelength. The two equations obtained by eliminating the pressure and spanwise velocity component depend on the variables X and Y denoting distance along and normal to the wall. These equations are parabolic in X, so that, given an initial disturbance imposed on the boundary layer at $X = \bar{X}$, a marching scheme can be set up to find the downstream development of that disturbance.

The non-parallel nature of the basic flow causes the usual problems when it comes to defining the condition of neutral stability (for a discussion of this problem in connection with Tollmien–Schlichting waves in boundary layers see e.g. Gaster 1974; Smith 1979). The flow quantity which we choose to monitor the growth or decay of a vortex is

$$E = \int_0^\infty (U^2 + V^2 + W^2)\,\mathrm{d}Y,$$

where U, V and W are the three dimensionless velocity components of the perturbation. The neutral position along the boundary layer is defined by the condition

$$\frac{\mathrm{d}E}{\mathrm{d}X} = 0.$$

In general we shall see that for walls of constant curvature a disturbance imposed sufficiently close to the leading edge of the wall gives a function $E(X)$ which is either always decreasing or increasing only over a finite range of values of X. The flow is taken to be unstable in the interval in which $E(X)$ is an increasing function of X.

At any position along the wall where $\mathrm{d}E/\mathrm{d}X = 0$, the local wavenumber a_x and Görtler number G_x can be calculated, and by varying the wavenumber of the initial disturbance a neutral curve can be generated in the (a_x, G_x)-plane. This neutral curve depends on both $\bar{X}$ and the form of the initial disturbance, and for $a_x \ll 1$ we find

$$G_x \sim a_x^{-2}, \tag{1.1}$$

whilst for $a_x \gg 1$ we obtain

$$G_x \sim a_x^4. \tag{1.2}$$

The above asymptotic forms are of course typical of convective or centrifugal instabilities, and (1.2) agrees well with the asymptotic results of I. In fact, as found in I, the right-hand branch of the neutral curve is not dependent on the form and location of the initial disturbance. However, when a_x decreases with the form of the initial disturbance fixed, the neutral curves for different values of $\overline{X}$ diverge and remain distinct even when (1.1) applies. By varying $\overline{X}$ we can obtain a critical value of G_x corresponding to the smallest minimum value of G_x on the sequence of neutral curves. In principle, by then varying the form of the initial disturbance, an overall critical value of G_x could be computed. Such a series of calculations would be prohibitively expensive, but an order of magnitude for the critical Görtler number can be inferred from a finite number of calculations.

Indeed, the main result of the present calculation is that the growth of Görtler vortices depends crucially on how the boundary layer is disturbed. Thus we show that there is no reason why a Görtler vortex triggered by a particular disturbance at some value of $\overline{X}$ should grow at the same value of the local Görtler number at which the same disturbance introduced elsewhere should begin to grow. This result is entirely consistent with the available experimental results, but is of course not in the spirit of parallel-flow calculations.

The procedure of this paper is as follows. In §2 we formulate the appropriate partial differential system describing the linear stage in the development of a Görtler vortex imposed at some location in the boundary layer. In §3 we describe the finite-difference scheme which we have used to solve this system. In §4 we describe the results of our calculations and compare them with experimental observations.

2. Formulation of the problem and some preliminary results

We consider the stability of a high-Reynolds-number viscous incompressible flow over a concave wall with radius of curvature $R\kappa(x/l)$, where l is a typical lengthscale along the wall. The variable x denotes distance measured along the wall whilst y denotes distance measured normal to the surface. Finally the variable z is chosen such that x, y and z form a mutually orthogonal coordinate system.

Suppose next that U_0 is a typical free-stream velocity in the x-direction, and then write

$$R_{\mathrm{E}} = \frac{U_0 l}{\nu}.$$

If the curvature parameter δ is then defined by

$$\delta = \frac{l}{R},$$

we confine our attention to the limit $R_{\mathrm{E}} \to \infty$ with

$$G = \frac{2l}{R} R_{\mathrm{E}}^{\frac{1}{2}}$$

held fixed. The parameter G is the so-called Görtler number, which of course plays the same role as the Taylor number in the Taylor-vortex problem. In I the further limit $G \to \infty$ was taken and an asymptotic form of the neutral curve appropriate to that limit was derived.

It is now convenient to introduce dimensionless coordinates X, Y and Z given by

$$(X, Y, Z) = l^{-1}(x, R_{\mathrm{E}}^{\frac{1}{2}} y, R_{\mathrm{E}}^{\frac{1}{2}} z),$$

where it has been anticipated that any z-variation of the flow should be on the boundary-layer lengthscale. Such a scaling is also suggested by the available experimental results, which show that when Görtler vortices occur their wavelength is comparable to the boundary-layer thickness. The basic flow is taken to be of the form

$$\mathbf{u} = U_0(\bar{u}(X, Y), R_E^{-\frac{1}{2}}\bar{v}(X, Y))(1+O(R_E^{-\frac{1}{2}})), \tag{2.1}$$

where $\bar{u}(X, Y)$, $\bar{v}(X, Y)$ are given functions of X and Y. Subsequently $\bar{u}$ and $\bar{v}$ are taken to be the velocity components of a Blasius boundary layer, but for the moment it is not necessary to be specific about these functions. The basic flow given by (2.1) is then perturbed such that

$$\mathbf{u} = U_0(\bar{u}+U(X, Y, Z), R_E^{-\frac{1}{2}}[\bar{v}+V(X, Y, Z)], R_E^{-\frac{1}{2}}W(X, Y, Z))(1+O(R_E^{-\frac{1}{2}})),$$

and by substituting into the Navier–Stokes equations written in terms of X, Y and Z it can be shown that the linearized equations which determine U, V, W and the corresponding pressure perturbation P (scaled on $(\nu U_0/l)$ are

$$U_X+V_Y+W_Z = 0, \tag{2.2a}$$

$$U_{YY}+U_{ZZ}-VU_Y = \bar{u}U_X+\bar{u}_X U+\bar{v}U_Y, \tag{2.2b}$$

$$V_{YY}+V_{ZZ}-\kappa G\bar{u}V-P_Y = \bar{u}V_X+U\bar{v}_X+\bar{v}V_Y+V\bar{v}_Y, \tag{2.2c}$$

$$W_{YY}+W_{ZZ}-P_Z = \bar{u}W_X+\bar{v}W_Y. \tag{2.2d}$$

Here terms of relative order $R_E^{-\frac{1}{2}}$ have been neglected and G is taken to be $O(1)$. The Z-dependence above can be taken to be periodic, but such a simplification does not reduce the partial differential equation in question to ordinary differential equations since there is no rational reason why the X-derivatives acting on U, V and W should be set equal to zero (as has been done in investigations prior to I). However, a crucial point to note is that the X-derivatives in (2.2b–d) arise from the convective terms in the Navier–Stokes equations rather than from the viscous terms, and this fact enables us to solve (2.2) by a marching procedure.

It is interesting to note that (2.2) also govern the centrifugal instability of certain boundary layers which arise in triple-deck theory (see e.g. Stewartson 1981; Smith 1982). Thus if X, Y and Z and the velocity components of the disturbance are made dimensionless using the scales appropriate to the lower deck of a triple deck then (2.2) are again found to govern the centrifugal instability of the flow in the lower deck. In this case $\bar{u}$ and $\bar{v}$ have a more complicated structure and it remains to be seen whether solutions of (2.2) confined to the lower deck exist.

Suppose next that U, V, W and P take the form

$$(U, V, W, P) = (U(X, Y)\cos az,\ V(X, Y)\cos az,\ W(X, Y)\sin az,\ P(X, Y)\cos az),$$

where a is the wavenumber of the vortex and is of course independent of X and Y. For computational purposes it is convenient to eliminate P and W between (2.2a, c, d) to obtain an equation involving only U and V. This equation, together with (2.2b), produces the following pair of equations to determine U and V:

$$U_{YY}-a^2U-\bar{u}U_X-\bar{u}_XU-\bar{v}U_Y-\bar{u}_Y V = 0, \tag{2.3a}$$

$$\begin{aligned} V\{\bar{u}_{XYY}+a^4+a^2\bar{v}_Y\}+\bar{v}_X U_{YY}+\{\bar{u}_{XXY}+a^2\bar{v}_X+\kappa a^2G\bar{u}\}\, U \\ +\left\{\bar{u}_{YY}-\bar{u}\frac{\partial^2}{\partial Y^2}+a^2\bar{u}\right\} V_X+2\left\{\bar{u}_{XY}+\bar{u}_X\frac{\partial}{\partial Y}\right\} U_X \\ + V_{YYYY}-\bar{v}V_{YYY}-\{\bar{v}_y+2a^2\}\, V_{YY}+\{\bar{u}_{XY}+a^2\bar{v}\}\, V_Y = 0. \end{aligned} \tag{2.3b}$$

Here it has been assumed that the basic flow satisfies the equation of continuity

$$\bar{u}_X + \bar{v}_Y = 0,$$

and certain terms in (2.2*b*) involving X-derivatives acting on U have been eliminated using (2.3*b*).

The vortex is assumed to be confined to the boundary layer and must satisfy the no-slip condition at the wall. Thus U and V must satisfy the boundary conditions

$$U = V = \frac{\partial V}{\partial Y} = 0 \quad (Y = 0, \infty). \tag{2.4}$$

However, it is clear from (2.3) that these conditions are not sufficient if the X-derivatives in (2.3) are not to be set arbitrarily equal to zero. If the basic velocity component $\bar{u}$ is always positive in $0 < Y < \infty$ then it can be seen from (2.3) that the conditions needed to specify the problem completely for U and V are

$$U = \overline{U}(Y), \quad V = \overline{V}(Y) \quad (X = \overline{X}). \tag{2.5}$$

Having imposed these initial conditions at $X = \overline{X}$ it can be seen from (2.3) that V and U for $X > \overline{X}$ can be obtained by marching forward in the X-direction with (2.3*a*,*b*) respectively. The initial conditions (2.5) correspond to some vortex perturbation being imposed on the flow at a given position along the curved wall. The growth or decay of the vortex downstream of $\overline{X}$ is dependent on the flow parameters a, G and basic flow $(\bar{u}, \bar{v})$. It is interesting to note that in any parallel-flow approximation to (2.3) the conditions (2.5) could not be satisfied without the existence of some completeness result about the discrete and continuous eigenfunctions of the appropriate linear differential system. It is also of interest to note that $W(Y)$ cannot be specified arbitrarily at $\overline{X}$ since, given U and V at $X = \overline{X}$, (2.3*b*) can be used to compute $\partial U/\partial X$ at $\overline{X}$, and then W follows from the equation of continuity.

The functions $\overline{U}$ and $\overline{V}$ must of course be consistent with the boundary conditions (2.4), so that

$$\overline{U}(0) = \overline{V}(0) = \overline{V}'(0) = \overline{U}(\infty) = \overline{V}(\infty) = 0. \tag{2.6}$$

Further conditions are necessary if U and V are not to have singularities at $X = \overline{X}$, $Y = 0$. The appropriate conditions are found in the usual way (see e.g. Goldstein 1948) by expanding U and V in Taylor series in $X - \overline{X}$ and Y at that point. If U and V are to have to have such Taylor series it can easily be shown that $\overline{U}$, $\overline{V}$ must also satisfy

$$\overline{U}''(0) = 0, \quad \overline{U}'''(0) = a^2\overline{U}'(0), \quad \overline{V}^{\mathrm{iv}}(0) = 2a^2\overline{V}''(0). \tag{2.7}$$

We could of course solve (2.3) without $\overline{U}$, $\overline{V}$ satisfying (2.7), but the numerical solution near $X = \overline{X}$, $Y = 0$ would have relatively large errors since U and V then have singularities there.

Before deriving a more convenient form for (2.3) it is instructive to consider (2.3*b*) in the limit $Y \to \infty$. If we assume that $\bar{u} \to 1$, $\bar{v} \to \bar{v}_\infty(X)$ when $Y \to \infty$ then, for large Y, U satisfies

$$U_{YY} - a^2 U - U_X - \bar{v}_\infty U_Y = 0.$$

If U is to tend to zero when $Y \to \infty$, then for large y we see from (2.6) that

$$U \sim \exp\left\{-\frac{Y^2}{4X}\right\} \quad (Y \gg 1),$$

but a parallel-flow approximation ignoring the term $\bar{u}\,\partial/\partial X$ above would give

$$U \sim \exp\{-aY\} \quad (Y \gg 1). \tag{2.8}$$

Thus parallel-flow theory (or any approximation that drops the term U_x in (2.3a)) leads to an incorrect form for the structure of the disturbance for $Y \gg 1$. Moreover, for $a \ll 1$ the asymptotic form (2.8) shows that the disturbance is no longer confined to the boundary layer. It is in this limit, and also for $a = O(1)$, that the parallel-flow theories give quite different results. For large a it was shown in I that the disturbance is confined to an internal viscous layer located where $|\bar{u}\bar{u}_y|$ has a local maximum. In this case parallel-flow theory gives correctly the first term in an asymptotic expansion of the neutral value of G in powers of $a^{-\frac{1}{2}}$. However, in this limit the non-parallel-flow theory of I can be used to write down in closed form the neutral Görtler number in an asymptotic form involving $a^{-\frac{1}{2}}$. Thus, in the only regime where parallel-flow theory has any relevance, a more accurate asymptotic result is readily available.

For $a = O(1)$ it is clear from above that separable solutions of (2.3) of the type found in I are no longer available, and numerical integration of (2.3) is necessary. In view of the previous discussion it is to be expected that such a task is made easier by making the change of variables $(X, Y) \to (X, \eta)$, where $\eta = Y(2X)^{-\frac{1}{2}}$). If this transformation is made then (2.3) become

$$U_{\eta\eta} - 2X\bar{u}U_X = 2X\{\bar{u}_Y V + \bar{u}_X U + a^2 U\} + \{(2X)^{\frac{1}{2}}\bar{v} - \eta\bar{u}\}\,U_\eta, \tag{2.9a}$$

$$\begin{aligned}
&V_{\eta\eta\eta\eta} + 4X^2\left\{\bar{u}_{YY} + a^2\bar{u} - \frac{\bar{u}}{2X}\frac{\partial^2}{\partial\eta^2}\right\} V_X \\
&= -V_{\eta\eta\eta}\{(2X)^{\frac{1}{2}}\bar{v} - \eta\bar{u}\} + 2X\left\{2a^2 + \bar{v}_Y - \frac{\bar{u}}{X}\right\} V_{\eta\eta} \\
&\quad + 2\sqrt{2}\,X^{\frac{3}{2}}\left\{-a^2\bar{v} + \bar{v}_{YY} + \frac{\alpha^2\eta\bar{u}}{(2X)^{\frac{1}{2}}} + \frac{\eta\bar{u}_{YY}}{(2X)^{\frac{1}{2}}}\right\} V_\eta + 4X^2\{\bar{v}_{YYY} - a^4 - a^2\bar{v}_Y\}\,V \\
&\quad - 2X\left\{\bar{v}_X + \frac{2\eta\bar{v}_Y}{(2X)^{\frac{1}{2}}}\right\} U_{\eta\eta} + 2\sqrt{2}\,X^{\frac{1}{2}}\{\bar{u}_X + \eta\bar{u}_{XY}(2X)^{\frac{1}{2}}\}\,U_\eta \\
&\quad - 4X^2\{-\bar{v}_{XYY} + a^2\kappa G\bar{u} + a^2\bar{v}_X\}\,U + 8X^2\left\{\bar{v}_{YY} + \frac{\bar{v}_y}{(2X)^{\frac{1}{2}}}\frac{\partial}{\partial_\eta}\right\} U_X.
\end{aligned} \tag{2.9b}$$

All the calculations carried out were for the Blasius velocity profile

$$\bar{u} = f'(\eta), \quad \bar{v} = \frac{1}{(2X)^{\frac{1}{2}}}\{\eta f' - f\},$$

where f satisfies

$$f''' + ff'' = 0,$$
$$f(0) = f'(0) = 0, \quad f'(\infty) = 1.$$

The aim of the calculations was to determine how the initial disturbance imposed at $\bar{X}$ develops downstream of that location. In order to measure the growth or decay of the disturbance some flow quantity must be monitored as the disturbance develops with increasing X. After some experimentation it was decided to use the dimensionless energy E of the disturbance defined by

$$E = \int_0^\infty \{U^2(X, Y) + V^2(X, Y) + W^2(X, Y)\}\,\mathrm{d}Y.$$

This quantity depends only on X, and, rewriting the above integral in terms of an integral over η, we obtain

$$E = (2X)^{\frac{1}{2}}\int_0^\infty \{U^2(X,\eta)+V^2(X,\eta)+W^2(X,\eta)\}\,d\eta$$
$$= E_1(2X)^{\frac{1}{2}}.$$

Thus we can show that σ, the local growth rate of the energy, is given by

$$\sigma(X) = \frac{\partial E/\partial X}{E} = \frac{\partial E_1/\partial X}{E_1}+\frac{1}{2X}. \tag{2.10}$$

For a given disturbance the position X_N of neutral stability is defined by the condition $\sigma(X_N) = 0$. Thus in our calculations we fix a and the Görtler number G and march downstream until σ vanishes. Without any loss of generality the Görtler number G can be set arbitrarily by suitable choice of the reference lengthscale l. It was found convenient to take $G = 0.025$ in the calculation that we will describe in §3. We stress that other choices of G give identical results, but with the variable X rescaled. In order to interpret the results it is convenient to define the local wavenumber a_x and Görtler number G_x scaled on the local boundary-layer thickness by

$$a_x = aX^{\frac{1}{2}},$$
$$G_x = GX^{\frac{3}{2}}\kappa(X)$$

respectively.

In I it was shown that for $a_x \gg 1$ the flow is neutrally stable where

$$G_x \sim a_x^4,$$

so that $\kappa(X)$ must increase at least as quickly as $X^{\frac{1}{2}}$ if the vortex is to be unstable for $X \gg 1$. In particular, for walls of constant curvature, $G_x \sim a_x^3$ and the flow is stable for $X \gg 1$.

3. The numerical scheme

The partial differential equations (2.9) were integrated using finite-difference approximations with step lengths h and ϵ in the η- and X-directions respectively. We denote the values of U and V at $\eta = nh$, $X = \overline{X}+j\epsilon$ by U_n^j and V_n^j, where $0 \leqslant n \leqslant N$ so that 'infinity' in the η-direction has been approximated by $\eta = \eta_\infty = Nh$. We suppose that $\{U_n^j\}$, $\{V_n^j\}$ are known at the jth location and show how these quantities are advanced to $X = X_{j+1} = \overline{X}+[j+1]\epsilon$. We discretize the X-momentum equation $(2.9a)$ by writing

$$\frac{U_{n+1}^{j+1}-2U_n^{j+1}+U_{n-1}^{j+1}}{h^2}-\frac{2X_j\bar{u}_n^j U_n^{j+1}}{\epsilon} = -\frac{2X_j\bar{u}_n^j U_n^j}{\epsilon}+F_n^j \quad (1 \leqslant n \leqslant N-1), \tag{3.1}$$

where $\bar{u}_n^j$ denotes $\bar{u}$-evaluation at $\eta = nh$, $X = \overline{X}+j\epsilon$ whilst F_n^j denotes the right-hand side of $(2.9a)$ evaluated at the jth step. Thus for example the term $-\eta\bar{u}U_\eta$ is replaced by $-\frac{1}{2}n\bar{u}_n^j\{U_{n+1}^j-U_{n-1}^j\}$.

The implicit scheme (3.1) leads to a tridiagonal system of $N-1$ equations for the $N-1$ unknowns $U_1^{j+1}, \ldots, U_{N-1}^{j+1}$ after setting $U_0^{j+1} = U_N^{j+1} = 0$. Having advanced U to the $(j+1)$th step the following scheme is then used to advance the normal velocity component V.

Consider firstly the terms on the right-hand side of (2.9*b*), which, apart from the last one, we discretize in terms of $\{U_n^j\}$, $\{V_n^j\}$ using the central-difference formulae

$$y_0' = \frac{y_1 - y_{-1}}{2h}, \quad y_0'' = \frac{y_1 - 2y_0 + y_{-1}}{h}, \quad y_0''' = \frac{y_2 - 2y_1 + 2y_{-1} - y_{-2}}{2h^3}$$

to approximate the derivatives in the η-direction. The last term on the right-hand side of (2.9*b*) is replaced by

$$\frac{8X_j^2}{\epsilon}\left\{(\bar{v}_{YY})_n^j\,[U_n^{j+1} - U_n^j] + \frac{(\bar{v}_Y)_j^n}{2h(2X_j)^{\frac{1}{2}}}[U_{n+1}^{j+1} - U_{n-1}^{j+1} - U_{n+1}^j + U_{n-1}^j]\right\} \quad (1 \leqslant n \leqslant N-1),$$

which is known already since U has been calculated at the $(j+1)$th step. The left-hand side of (2.9*b*) is replaced by

$$\frac{1}{h^4}\{V_{n+2}^{j+1} - 4V_{n+1}^{j+1} + 6V_n^{j+1} - 4V_{n-1}^{j+1} + V_{n-2}^{j+1}\}$$
$$+\frac{4X_j^2}{\epsilon}\left\{(\bar{u}_{yy} + a^2\bar{u})_n^j\,[V_n^{j+1} - V_n^j] - \frac{(\bar{u})_n^j}{4X_j h^2}[V_{n+1}^{j+1} - 2V_n^{j+1} + V_{n+1}^{j+1} - V_{n+1}^j + 2V_n^j - V_{n-1}^j]\right\}$$
$$(1 \leqslant n \leqslant N-1).$$

Thus using the boundary condition to show that

$$V_0^j = V_N^j = V_{n+1}^j = 0, \quad V_{-1}^j = V_1^j,$$

we obtain a pentadiagonal system of $N-1$ equations for $V_1^{j+1}, \ldots, V_{N-1}^{j+1}$. Having solved this system we have then advanced both U and V to the $(j+1)$th step, and the process can be repeated to find U and V at the $(j+2)$th step, etc.

Numerical investigations of the above scheme confirmed that the evaluation of $U_{\eta\eta}$ and $V_{\eta\eta\eta\eta}$ in (2.9*a*, *b*) at the jth step provides a scheme which is stable for $\epsilon = O(h)$. Prior to using this scheme, these derivatives were evaluated at the jth step, but the resulting scheme proved to be stable only for $\epsilon = O(h^2)$. These results are of course typical of diffusion equations with constant coefficients; in our problem the ratio of ϵ/h required for stability will depend on X, and our checks indicated that in general the smallest value of ϵ/h was required near $\bar{X}$, where, as we shall see, the flow changes relatively quickly.

In order to evaluate the energy integral E it is of course necessary for us to compute the spanwise velocity component W. If the continuity equation is written in terms of X and η we obtain

$$-aW = \frac{\partial U}{\partial X} - \frac{\eta}{2X}\frac{\partial U}{\partial H} + \frac{1}{(2X)^{\frac{1}{2}}}\frac{\partial V}{\partial \eta},$$

so that to calculate W at the jth step U_x, U and V are required. If U and V are known at the jth step, then U_x can be evaluated from (2.9*a*) by writing

$$U_x = \frac{1}{2X\bar{u}}\{U_{\eta\eta} - 2X[\bar{u}_y V + \bar{u}_x U + a^2 U] - [(2X)^{\frac{1}{2}}\bar{v} + \eta\bar{u}]\,U_\eta\}$$

and then using central differences to approximate the η-derivatives appearing on the right-hand side of this equation. Alternatively U_x can be calculated by using $\{U_n^j\}$, $\{U_n^{j+1}\}$ and writing

$$\frac{\partial U}{\partial X}(\eta = nh,\, X = \bar{X} + j\epsilon) = \frac{U_n^{j+1} - U_n^j}{\epsilon}.$$

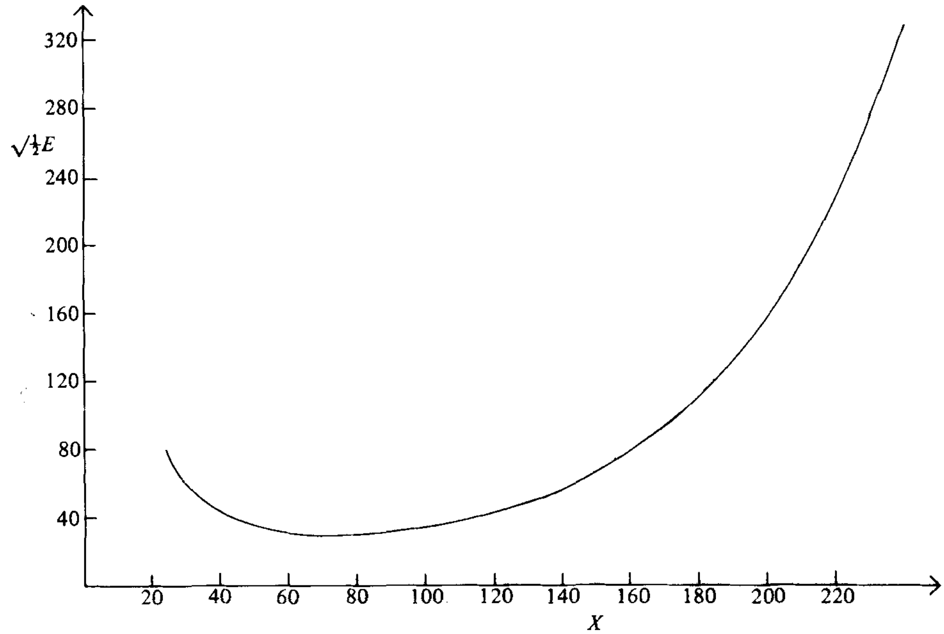

FIGURE 1. The development of E with increasing X, corresponding to the initial conditions (3.1) and $a = 0.069$, $G = 0.025$ and $X = 20$.

Both methods were used and gave identical results, which gives a check on the implicit scheme used to advance U to the $(j+1)$th step.

Having calculated $\partial U/\partial X$, the spanwise velocity component can be calculated and the energy integral computed using Simpson's rule.

We now describe some of the checks which were used to determine the accuracy of the numerical scheme described above. The disturbance imposed on the flow was taken to be

$$U = \eta^6 e^{-\eta^2}, \quad V = 0, \quad X = \bar{X} = 20, \tag{3.2}$$

which of course satisfies the required conditions at $Y = 0$. The parameters a and G have the values 0.069 and 0.025 respectively, whilst $\kappa = 1$ and infinity was approximated by $\eta = \eta_\infty = 10$. With a fixed value of $h = 0.1$ several calculations were performed using different values of ϵ/h and the stability of the scheme was confirmed for $\epsilon = O(h)$. In fact it was found that $\epsilon/h = 2$ was sufficiently accurate for our purposes. Further checks were performed by varying either η_∞ or h with ϵ/h fixed, and it was determined that $h = 0.1$, $\eta_\infty = 10$ give sufficiently accurate results. Apart from the checks described above, several other test runs corresponding to different initial conditions with different values of a and G were carried out, and it was not found necessary to change the values of ϵ, h, and η_∞ given above.

4. Results and discussion

We describe finally our results obtained by integrating the disturbance equations with the initial data as given by (3.2) and with $(\eta_\infty, h, \epsilon, a, G, \kappa) = (10, 0.1, 0.2, 0.069, 0.025, 1)$. In figures 1 and 2 we have shown how the energy and growth rate $\sigma(X)$ vary downstream of $X = \bar{X}$, where the disturbance is introduced into the flow. Immediately downstream of $\bar{X}$ the energy decreases until it reaches a minimum at

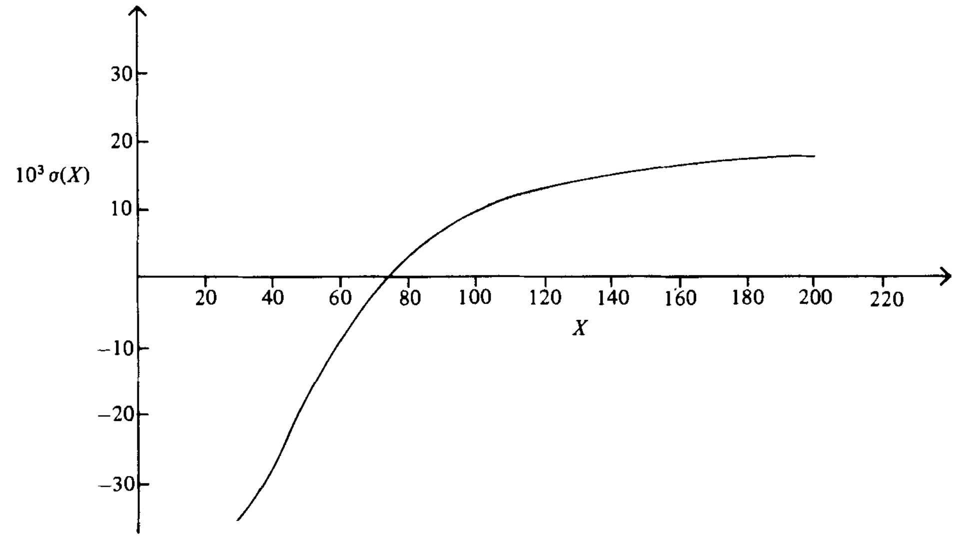

FIGURE 2. The growth-rate function $\sigma(X)$ corresponding to the initial conditions (3.1) and $a = 0.069$, $G = 0.025$ and $\bar{X} = 20$.

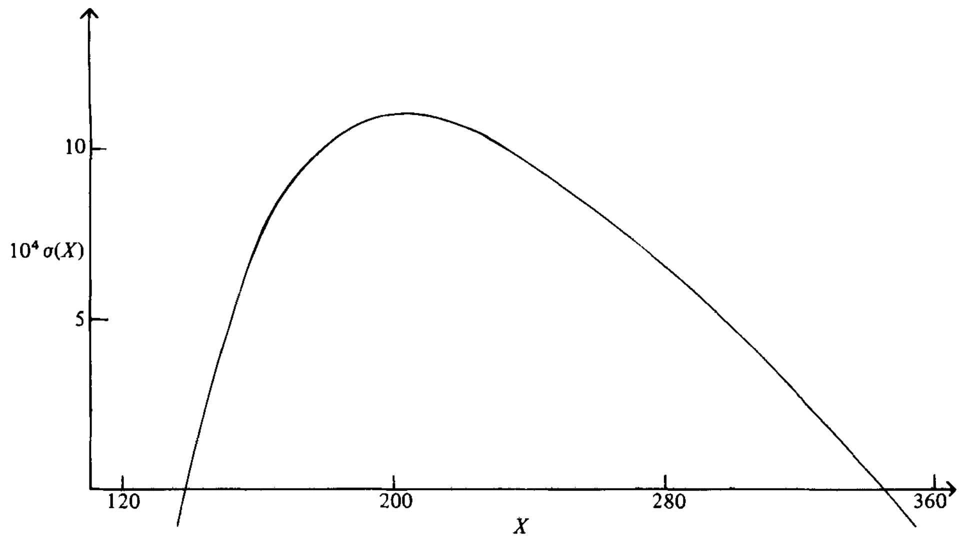

FIGURE 3. The growth-rate function $\sigma(X)$ corresponding to the initial conditions (3.1) and $a = 0.1081$, $G = 0.025$ and $\bar{X} = 50$.

$X \approx 74$, beyond which the local growth rate $\sigma(X)$ is positive and the flow is said to be unstable. In fact $\sigma(X)$ increases until it reaches a maximum, after which it decreases monotonically and becomes negative again where $X \approx 10^4$. Thus the particular Görtler vortex introduced into the flow at $\bar{X}$ is unstable only for a finite range of values of X. It is to be expected that for a wall of constant curvature a given vortex is always stable when $X \to \infty$ because the local wavenumber a_x and the local Görtler number G_x satisfy $G_x \sim a_x^3$, whereas from I we know that for $a_x \gg 1$ the neutral value of G_x is $O(a_x^4)$.

The calculation described above was repeated with $a = 0.1081$ and $\bar{X} = 50$ to

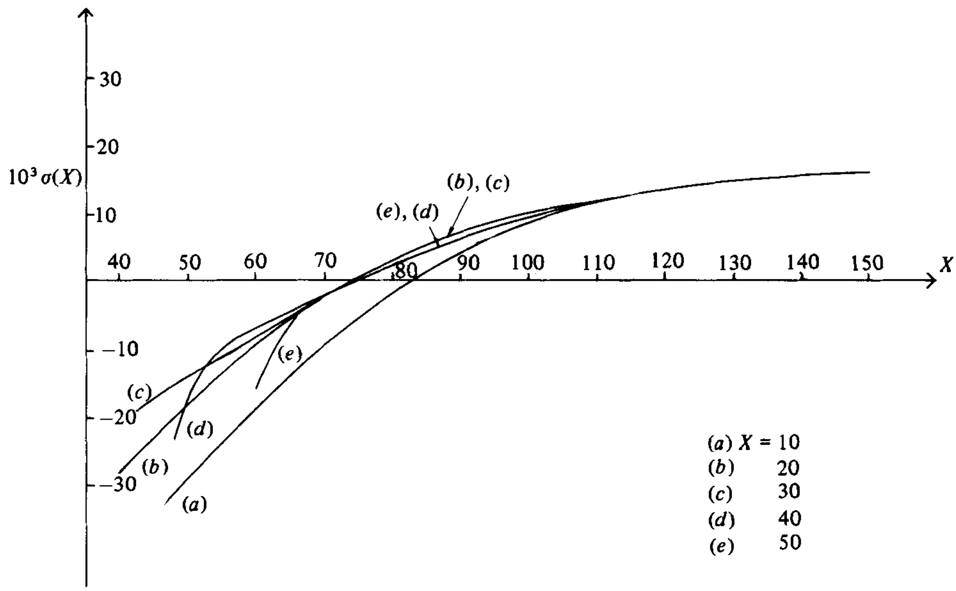

FIGURE 4. The dependence of $\sigma(X)$ on the location of the initial disturbance, with $a = 0.069$, $G = 0.025$.

produce the results shown in figure 3. If $\bar{X}$ is held fixed and a increased then the range of values for which $\sigma(X) > 0$ decreases until at $a = a^*$ the growth-rate function $\sigma(X)$ and its derivative vanish simultaneously. If the calculation is repeated for $a > a^*$ then $\sigma(X)$ remains negative for all X. The value of a^* depends on $\bar{X}$ and the form of the initial perturbation.

In figure 4 we have shown how the function $\sigma(X)$ changes when $\bar{X}$, the location of the initial disturbance, is varied, with $a = 0.069$. It can be seen from this figure that ultimately all the growth rate curves merge when X increases and that, apart from the case $\bar{X} = 10$, the neutral positions where $\sigma(X) = 0$ lie very close together. If $\bar{X}$ is decreased below $\bar{X} = 10$, the zeros of $\sigma(X)$ move progressively to the right, so that the flow remains stable over an increasingly large range of values of the Görtler number. We know of no physical reason why, as the latter result suggests, disturbances introduced progressively closer to the leading edge should be increasingly less efficient in triggering the downstream linear growth of Görtler vortices.

We now turn to the manner in which the velocity field changes as the boundary layer grows downstream of $X = \bar{X}$. We have illustrated in figures 5(a, b, c) how U, V and W, corresponding to the results of figures 1, 2, vary with X. We recall that initially the normal velocity component of the initial disturbance is zero and it can be seen that the size of this component increases monotonically with X. The spanwise velocity component W is initially proportional to $\partial U/\partial X$, which is non-zero. Thus U and W both decay initially downstream of $\bar{X}$ before growing at larger values of X. It is clear from figure 5 that the essential shape of the three velocity components does not change greatly with increasing X. We also point out that since $\eta = Y(2X)^{-\frac{1}{2}}$ the vortices diffuse outwards at the same rate as does the basic Blasius profile. Moreover, since the wavelength of the vortices remains constant with X increasing, the vortices become more elongated in the Y-direction as they develop downstream of $\bar{X}$. It is also clear that the development discussed above could not be predicted by a parallel flow theory which ignores X-derivatives in the linear stability equations (2.9).

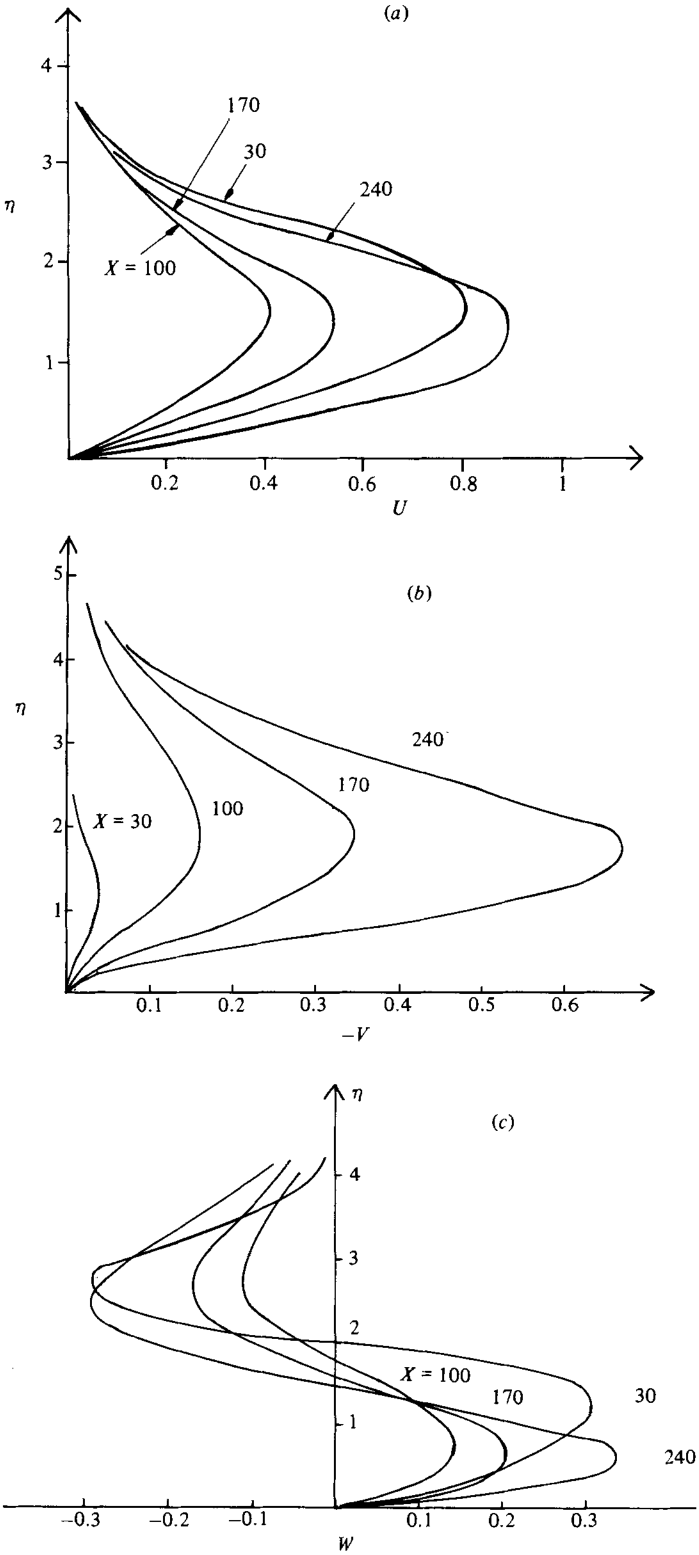

FIGURE 5. The development of the velocity field corresponding to the calculations performed in order to construct figures 1, 2.

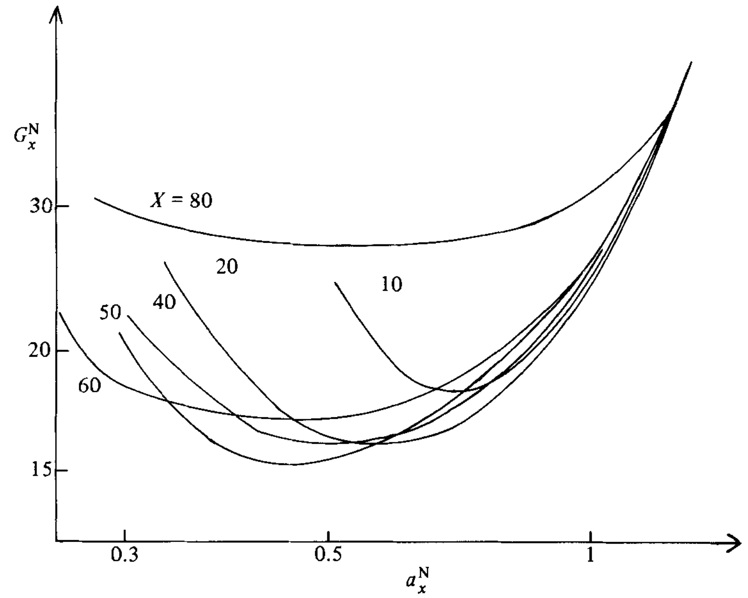

FIGURE 6. The neutral curves corresponding to different locations of the initial conditions (3.1), with $G = 0.025$.

We have found then that, depending on the wavenumber a, the growth-rate function $\sigma(X)$ has at most two zeros. At any value of X where $\sigma(X)$ vanishes, the local wavenumber a_x and Görtler number G_x can be calculated and are denoted by a_x^N, G_x^N, respectively. If $\bar{X}$ is held fixed and a is varied, a neutral curve in the (a_x^N, G_x^N)-plane can be drawn on which $\sigma(X)$ vanishes. The results of such a calculation for several values of $\bar{X}$ are illustrated in figure 6. Each neutral curve has a minimum, but, whereas the right-hand branches of these curves ultimately merge, the left-hand branches remain distinct. In fact, as we shall see shortly, the right-hand branches approach the asymptotic neutral curve given in I, whilst on the left-hand branches the results imply that, when $a_x^N \to 0$, $G_x^N \sim (a_x^N)^{-2}$ for any given $\bar{X}$. We can also infer from figure 6 that there is a value of $\bar{X}$, close to $\bar{X} = 50$, which leads to a growing Görtler vortex at the least value of the Görtler number G_x^N. If $\bar{X}$ is increased, then, surprisingly, the neutral curves become much more flat, with a minimum value of G_x^N always greater than G_x corresponding to $X = \bar{X}$. Thus however far downstream the flow is disturbed, $\sigma(X)$ is always initially negative and the flow is locally stable. This result should perhaps not be entirely unexpected, since an arbitrary initial disturbance will in general contain components that decay immediately downstream of $\bar{X}$ with large damping rates. If calculations are performed for values of $\bar{X} < 10$, it is found that the neutral curves follow the trend indicated by the curves corresponding to $\bar{X} = 10$, 20 and more progressively up and across to the right. We have no physical explanation why this should be the case.

Apart from the calculations reported above for the initial conditions (3.11), several different forms for the initial conditions were investigated. The qualitative dependence of the resulting neutral curves on $\bar{X}$ was found to be quite similar to that described above. In figure 7 we have shown the neutral curves corresponding to several different forms of initial condition imposed at $\bar{X} = 50$. In this figure we have also plotted the two-term asymptotic neutral curve

$$G_x^N = 5.91(a_x^N)^4\left\{1+\frac{0.96}{a_x^N}\right\}$$

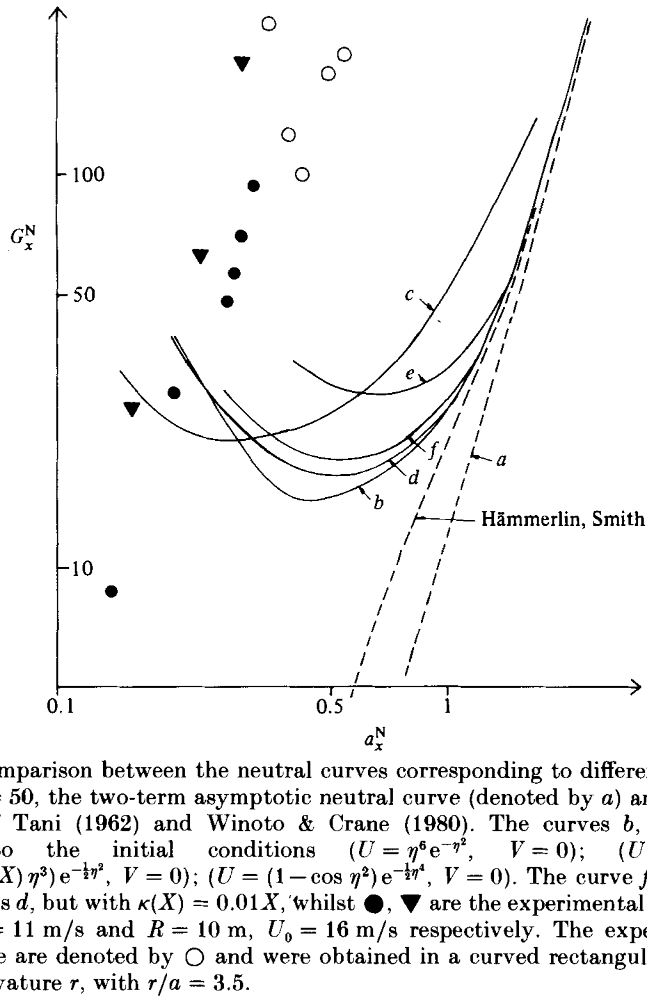

FIGURE 7. A comparison between the neutral curves corresponding to different initial conditions imposed at $\overline{X} = 50$, the two-term asymptotic neutral curve (denoted by a) and the experimental observations of Tani (1962) and Winoto & Crane (1980). The curves b, c, d, e correspond respectively to the initial conditions ($U = \eta^6 e^{-\eta^2}$, $V = 0$); ($U = 0$, $V = \eta^5 e^{-\eta^3}$); $V = (\eta + (\frac{1}{2} + \frac{1}{3}a^2 X)\eta^3) e^{-\frac{1}{2}\eta^2}$, $V = 0$); ($U = (1 - \cos \eta^2) e^{-\frac{1}{2}\eta^4}$, $V = 0$). The curve f corresponds to the initial conditions d, but with $\kappa(X) = 0.01X$, whilst ●, ▼ are the experimental results of Tani with $R = 10$ m, $U_0 = 11$ m/s and $R = 10$ m, $U_0 = 16$ m/s respectively. The experimental results of Winoto & Crane are denoted by ○ and were obtained in a curved rectangular channel of width a, radius of curvature r, with $r/a = 3.5$.

found in I. The agreement between the asymptotic result and the numerical results is satisfactory even for moderate values of a_x^N, even though the asymptotic result is valid only in the limit $a_x^N \to \infty$.

The neutral curves shown in figure 7 change when the location $\overline{X}$ of the initial disturbance is varied. We could, of course, find the most dangerous location for each type of disturbance by varying $\overline{X}$ and plotting the corresponding neutral curves. Such a large calculation would show which of the finite set of initial conditions under consideration produces a growing vortex at the smallest Görtler number. There is, of course, no reason to suppose that even more dangerous critical conditions could not be found by varying $\overline{U}$ and $\overline{V}$, so it was decided not to perform that calculation. However, our calculations with various initial conditions at different values of $\overline{X}$ suggest that the most dangerous possible initial conditions would lead to a neutral curve with a minimum value of G_x^N the same order of magnitude as the smallest minimum value of G_x^N shown in figure 7.

Our principal conclusion, therefore, is that the concept of a unique neutral curve, so familiar in hydrodynamic stability theory, is not tenable in the Görtler-vortex problem if the growth of the boundary layer is taken into account in a self-consistent manner. We predict therefore that experimentally the location where a Görtler vortex will appear in a boundary layer will depend on how the disturbances responsible for the growth of the vortices arise. Some experimental results consistent with this view have been given by Winoto & Crane (1980).

We now turn to the question of how our numerical results are related to previous parallel-flow approximations and experimental observations. In figure 7 we have shown, for example, the parallel-flow theories of Hammerlin (1956) and Smith (1955). It is clear that the parallel-flow theories have any relevance only at high wavenumbers, where, as shown in I, an asymptotic neutral curve is easily generated. When the wavenumber decreases, the different parallel-flow theories give increasingly divergent neutral curves, until at small wavenumbers the various parallel-flow theories are in complete disagreement. This disagreement between the parallel-flow theories is, of course, not related to the non-uniqueness of the neutral curve that we have found. In the parallel-flow theories the initial-value problem is not considered, and the wide spread of the neutral curves at small wavenumbers results from the inability of any parallel-flow approximation to describe correctly the decay of a vortex at the edge of the boundary layer.

In figure 7 we have also shown some experimentally determined points due to Tani (1962) and Winoto & Crane (1980). It should be noted that the neutral curves we have plotted correspond to a particular form for the initial condition, and neutral curves below the ones shown can be found by varying $\overline{U}$ and $\overline{V}$. Nevertheless, apart from the experimental point in the lower left-hand corner of this figure, the experimental results are consistent with our calculations. Indeed, since at finite wavenumbers a unique neutral curve does not exist, the agreement between theory and experiment shown in figure 7 is perhaps all that we can reasonably expect in the Görtler problem. The single experimental point in the lower left-hand corner of figure 7 could possibly correspond to some initial condition more complicated than those considered in this paper. It should be stressed that the different experimental points corresponding to a particular experimental run describe the variation of the wavenumber and Görtler number of a vortex downstream of where it is first observed. Thus, since the boundary layer grows like $X^{\frac{1}{2}}$, the local Görtler number and wavenumber grow, respectively, like $X^{\frac{3}{4}}$ and $X^{\frac{1}{2}}$ as the disturbance moves downstream, so that the experimental points corresponding to a particular run should be on a line of slope $\frac{3}{2}$.

We now turn to the question of whether the left-hand branches of the numerically obtained neutral curves have a definite asymptotic structure. We recall that Smith (1979, 1980) has shown that Tollmien–Schlichting waves in growing boundary layers have two distinct asymptotic structures to the neutral curve at high Reynolds numbers. Thus we might expect that when $a_x^{\mathrm{N}} \to 0$ an asymptotic structure should be available to complement the large-a_x^{N} structure of I. On the basis of what is known about the related Bénard and Taylor problems it is to be expected that when $a_x^{\mathrm{N}} \to 0$ the Görtler number will be of order $(a_x^{\mathrm{N}})^{-2}$ on the left-hand branch. An investigation of the numerical results that we have obtained confirms that on the left-hand branches of, for example, the curves in figure 6, $G_x^{\mathrm{N}} \sim (a_x^{\mathrm{N}})^{-2}$ for sufficiently small a_x^{N}.

The latter result suggests that for $a \ll 1$ an asymptotic solution of (2.9) can be found by first expanding G in the form

$$G = \frac{g_0}{a^2} + \frac{g_1}{a} + \dots$$

and then writing down similar expansions for U and V with $u = O(V)$. If these expansions are substituted into (2.9) and like powers of a are equated, then at first order we obtain (2.9) with $a = 0$ and a^2G replaced by g_0. Thus, in contrast with the high-wavenumber limit, the small-wavenumber limit leads to a sequence of partial differential equations to be solved. The coefficient g_0 is then fixed and the value of X obtained at which the flow is locally neutrally stable. This position will depend on the form of the imposed initial conditions, so that the left-hand branch of the neutral curve is not unique. Thus it is only at large wavenumbers that a unique neutral curve exists and parallel-flow theory has any validity.

We now make some comments on the possible relevance of this work to the transition process for boundary layers on curved walls. We note that Bippes & Görtler (1972) have described an experimental investigation of this process, and we shall describe some of their results. It was found by Bippes & Görtler that, wherever vortices develop, they are initially steady and only become time dependent some distance downstream of where they first develop. This time dependence occurs when the vortices become unstable to wavy-vortex-like perturbations after first growing into finite-amplitude steady Görtler vortices. In I some discussion of vortices that propagate in the flow direction was given, and it was found that at high wavenumbers such disturbances are more stable than steady disturbances, and there is no reason to suppose that this is not the case at finite wavenumbers. Moreover, since the experiments show no evidence of the vortices being time dependent in the linear regime, we feel that our assumption of steady perturbation is sensible.

In practical situations we expect that both Tollmien–Schlichting waves and Görtler vortices will be important in the transition process on curved walls. The interaction between these modes of instability is a non-trivial problem, but we can make some tentative remarks about the circumstances under which empirical laws such as the e^9 law would be dominated by Görtler vortices. Our calculations have shown that for Görtler numbers $O(10)$ the growth rates of Görtler vortices are an order of magnitude larger than those typical of Tollmien–Schlichting waves. This suggests that, for flows with Görtler numbers greater than about 10, the prediction of transition based on, say, the e^9 law, would be dominated by the effect of Görtler vortices. In view of the definition of the Görtler number, and the fact that Tollmien–Schlichting waves are unstable for $R_E^{\frac{1}{2}} \sim O(250)$, it is to be expected that Görtler vortices dominate the transition process for walls with $l/R > O(\frac{1}{50})$.

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